"Mercuriale de groupes et de relations"
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Question 1: Let α, β be free, ergodic, measure preserving actions $\mathbb{F}_n \curvearrowright^{\alpha} (X, \mu)$ and $\mathbb{F}_m \curvearrowright^{\beta} (X, \mu)$.
If α, β produce the same orbits, must $n = m$?

Gaboriau '98
↳ Yes!

Question 2: What about $\mathbb{Z}^n$ & $\mathbb{Z}^m$?

Theorem (Dye '59 for $\mathbb{Z}$, Ornstein–Weiss '80 for amenable) Any two free ergodic m.p. actions of amenable groups are orbit equivalent.

$\Longrightarrow$ answer to Q2 is no!

- E a countable Borel equivalence relation on (X, μ)
- E is probability measure preserving if for any Borel automorphism γ that permutes the E-classes, γ is pmp
$\mu(\gamma(A)) = \mu(A)$
- G is a graphing of E if $E = E_G$
- T is a treeing of E if T is an acyclic graphing.

Cost = "# edges"

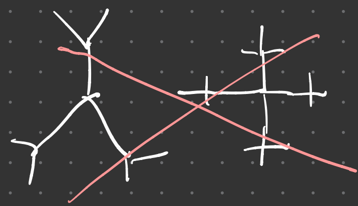
For a loc. countble pmp G

$$\mathrm{cost}_\mu(G) := \frac{1}{2}\int |G_x| \, d\mu = \int |\vec{G}_x| \, d\mu$$

$$\mathrm{cost}_\mu(E) := \inf\{\mathrm{cost}_\mu(G) : G \text{ is a Borel graphing}\}$$

Theorem (Gaboriau '98)

Treeings achieve cost.
(T Borel treeing of E
$\Rightarrow \mathrm{cost}_\mu(E) = \mathrm{cost}_\mu(T)$)



(a) T has bounded degree ($\le d$)
(b) $\exists L, M > 0$ s.t.
$$\frac{1}{M} d_T \le d_G \le L \, d_T$$

Euler "# edges − # vertices"

$$\mathrm{Euler}_\mu(G) := \mathrm{cost}_\mu(G) - \mu(X)$$

Proof Fix a graphing G of E. Want to show $\mathrm{cost}_\mu(G) \geq \mathrm{cost}_\mu(T)$.

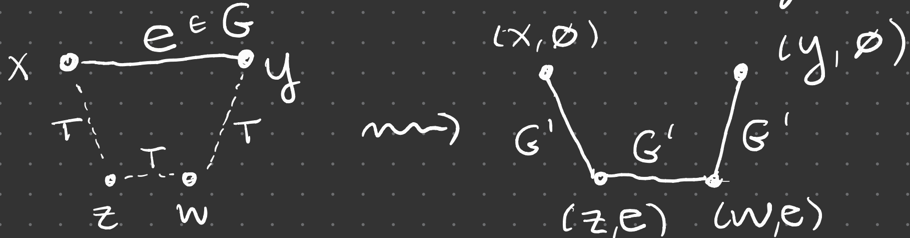
Assume that:

(a) T has bounded degree $(\leq d)$

(b) $\exists L, M > 0$ s.t.

$$\frac{1}{M} d_T \leq d_G \leq L\, d_T$$

Idea 1) "Implement" G via edges of T



$X' := X \sqcup \{(x,e) : x$ lies on the interior of the T-path connecting $e\}$

$G' :=$ corresponding T edges

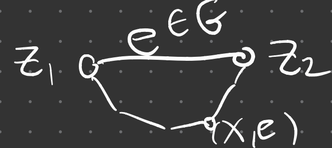
Euler "# edges − # vertices"

$$\mathrm{Euler}_\mu(G) := \mathrm{cost}_\mu(G) - \mu(X)$$

$$\mu'(A) := \int_X |\mathrm{proj}^{-1}(x) \cap A| \, d\mu$$

$\star$ μ' is still finite: $\mathrm{proj}(y) = x \to y = (x, e)$

$\leq d^{2M}$ possibilities for endpoints



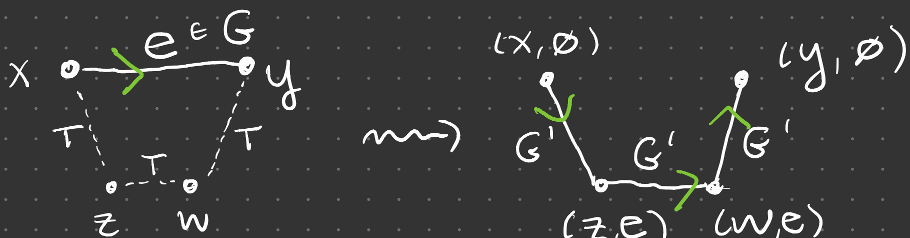
★ $\mu'|_X = \mu$

$$\mathrm{Euler}_{\mu'}(G') = \mathrm{Euler}_{\mu}(G)$$

$$\mathrm{cost}_{\mu'}(G') - \mu'(X') = \mathrm{cost}_{\mu}(G) - \mu(X)$$

$$\mathrm{cost}_{\mu'}(G') = \mathrm{cost}_{\mu}(G) + \mu'(X' \setminus X)$$

Fix a Borel directing of G, $\vec{G}$ and direct G' accordingly



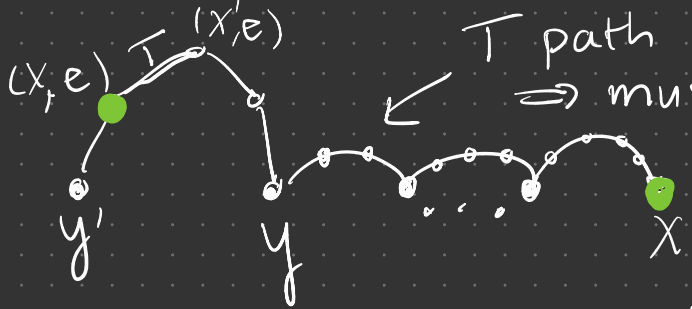
$\Rightarrow$ each fake vertex has outdeg $= 1$
& the outdeg of each true vertex is the same

$$\mathrm{cost}_{\mu'}(G') = \int_{X'} |\vec{G}'_x| \, d\mu'$$

$$= \int_X |\vec{G}'_x| \, d\mu' + \int_{X' \setminus X} |\vec{G}'_x| \, d\mu'$$

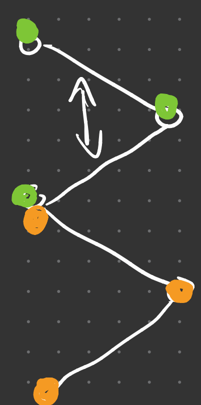
$$= \mathrm{cost}_{\mu}(G) + \mu'(X' \setminus X) \checkmark$$

Idea 2: If (x,e) is a fake vertex, look at true vertex $(x,\emptyset)$:



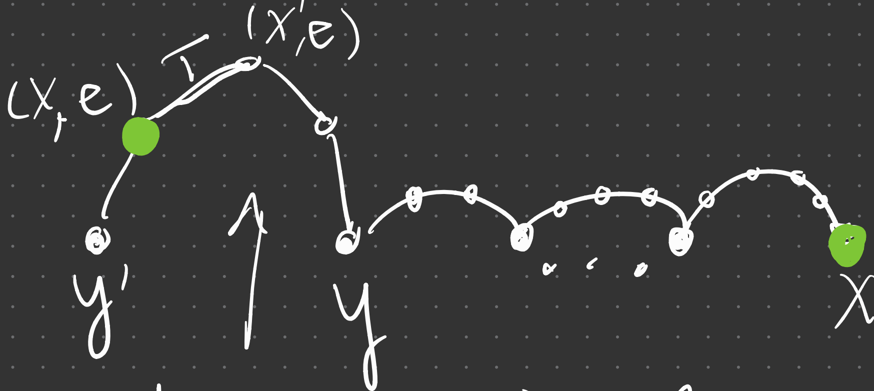
T path from x to x $\Rightarrow$ must have a back-tracking $z \to w \to z$

Countably color triples of points to be folded (so that if two triples are distance $\le d^{2M}$, get different colors)



$Y_0 := X'$ $H_0 := G'$

Y_{m+1}, H_{m+1} by folding triples of color m, choosing which vertex survives so that true vertices always survive.



$$d_{G'}((x,e),(y,\emptyset)) \le M$$

$$d_T(x,y) \le M$$

$$d_G(x,y) \le LM$$

$$d_{G'}(x,y) \le LM^2$$

$$d_{G'}((x,e),(x,\emptyset)) \le M + LM^2$$

after iterating (X_n, G_n) $\le M+LM^2$ many times, $X_{M+2M^2} = X$

$$G_{M+LM^2} = T$$

$$\begin{aligned} Euler_\mu(G) &= Euler_{\mu'}(G') \\ &\geq Euler_{\mu'}(G_{M+LM^2}) \\ &= Euler_\mu(T) \end{aligned}$$

$$\mathrm{cost}_\mu(G) - \cancel{\mu(X)} \geq \mathrm{cost}_\mu(T) - \cancel{\mu(X)}$$